

# GLOBAL EXISTENCE OF WEAK SOLUTIONS TO DISSIPATIVE TRANSPORT EQUATIONS WITH NONLOCAL VELOCITY

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ABSTRACT. We consider 1D dissipative transport equations with nonlocal velocity field:

$$\theta_t + u\theta_x + \delta u_x\theta + \Lambda^\gamma\theta = 0, \quad u = \mathcal{N}(\theta),$$

where  $\mathcal{N}$  is a nonlocal operator given by a Fourier multiplier. Especially we consider two types of nonlocal operators:

- (1)  $\mathcal{N} = \mathcal{H}$ , the Hilbert transform,
- (2)  $\mathcal{N} = (1 - \partial_{xx})^{-\alpha}$ .

In this paper, we show several global existence of weak solutions depending on the range of  $\gamma$  and  $\delta$ . When  $0 < \gamma < 1$ , we take initial data having finite energy, while we take initial data in weighted function spaces (in the real variables or in the Fourier variables), which have infinite energy, when  $\gamma = 1$ .

## 1. INTRODUCTION

In this paper, we consider transport equations with nonlocal velocity. Here, the non-locality means that the velocity field is defined through a nonlocal operator that is represented in terms of a Fourier multiplier. For example, in the two dimensional Euler equation in vorticity form,

$$\omega_t + u \cdot \nabla \omega = 0,$$

the velocity is recovered from the vorticity  $\omega$  through

$$u = \nabla^\perp (-\Delta)^{-1} \omega \quad \text{or equivalently} \quad \hat{u}(\xi) = \frac{i\xi^\perp}{|\xi|^2} \hat{\omega}(\xi).$$

Other nonlocal and quadratically nonlinear equations appear in many applications. Prototypical examples are the surface quasi-geostrophic equation, the incompressible porous medium equation, Stokes equations, magneto-geostrophic equation in multi-dimensions. For more details on nonlocal operators in these equations, see [1].

We here study 1D models of physically important equations. The 1D reduction idea were initiated by Constantin-Lax-Majda [8]: they proposed the following 1D model

$$\theta_t = \theta \mathcal{H} \theta$$

for the 3D Euler equation in the vorticity form and proved that  $\mathcal{H}\theta$  blows up in finite time under certain conditions. Motivated by this work, other similar models were proposed and analyzed in the literature [1, 2, 3, 4, 5, 6, 12, 13, 16, 19, 20, 23]. In this paper, we consider the following 1D equation:

$$\theta_t + u\theta_x + \delta u_x\theta + \nu\Lambda^\gamma\theta = 0, \quad u = \mathcal{N}(\theta). \quad (1.1)$$

Depending on a nonlocal operator  $\mathcal{N}$ , (1.1) has structural similarity of several important fluid equations as described below. The goal of this paper is to show the existence of weak solutions with rough initial data. To this end, we will choose functionals carefully to extract more information from the structure of the nonlinearity to construct weak solutions.

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1.1. **The case  $\mathcal{N} = \mathcal{H}$ .** We first take the case  $\mathcal{N} = \mathcal{H}$ , the Hilbert transform. Then, (1.1) becomes

$$\theta_t + (\mathcal{H}\theta)\theta_x + \delta\theta\Lambda\theta + \nu\Lambda^\gamma\theta = 0, \quad (1.2)$$

where the range of  $\gamma$  and  $\delta$  will be specified below. We note that (1.2) is considered as an 1D model of the dissipative surface quasi-geostrophic equation. The surface quasi-geostrophic equation describes the dynamics of the mixture of cold and hot air and the fronts between them in 2 dimensions [10, 26]. The equation is of the form

$$\theta_t + u \cdot \nabla\theta + \nu\Lambda^\gamma\theta = 0, \quad u = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \quad (1.3)$$

where the scalar function  $\theta$  is the potential temperature and  $\mathcal{R}_j$  is the Riesz transform

$$\mathcal{R}_j f(x) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{(x_j - y_j)f(y)}{|x - y|^3} dy, \quad j = 1, 2.$$

As Constantin-Lax-Majda did for the Euler equation, the equation (1.2) is derived by replacing the Riesz transforms with the Hilbert transform. The case  $\delta = 0$  and  $\delta = 1$  correspond to (1.3) in non-divergence and divergence form, respectively. We take a parameter  $\delta \in [0, 1]$  to cover more general nonlinear terms in (1.2). We note that there are several singularity formation results when  $\nu = 0$ :  $0 < \delta < \frac{1}{3}$  and  $\delta = 1$  [23],  $0 < \delta \leq 1$  [6], and  $\delta = 0$  [12, 19, 28]. By contrast, we look for weak solutions of (1.2) globally in time (see e.g. [14]). From now on, we set  $\nu = 1$  for notational simplicity.

We assume that  $\theta_0$  satisfies the conditions

$$\theta_0(x) > 0, \quad \theta_0 \in L^1 \cap H^{\frac{1}{2}}. \quad (1.4)$$

Since (1.2) satisfies the minimum principle (see Section 2) when  $\delta \geq 0$ ,  $\theta(t, x) \geq 0$  for all time. Moreover, the structure of the nonlinearity enables us to use the following function space

$$\mathcal{A}_T = L^\infty \left(0, T; L^p \cap H^{\frac{1}{2}}\right) \cap L^2 \left(0, T; H^{\frac{\gamma+1}{2}}\right) \quad \text{for all } p \in (1, \infty).$$

**Definition 1.1.** We say  $\theta$  is a weak solution of (1.2) if  $\theta \in \mathcal{A}_T$  and (1.2) holds in the following sense: for any test function  $\psi \in \mathcal{C}_c^\infty([0, T) \times \mathbb{R})$ ,

$$\int_0^T \int_{\mathbb{R}} [\theta\psi_t + (\mathcal{H}\theta)\theta\psi_x + (1 - \delta)\Lambda\theta\theta\psi - \theta\Lambda^\gamma\psi] dxdt = \int_{\mathbb{R}} \theta_0(x)\psi(0, x)dx$$

holds for any  $0 < T < \infty$ .

**Theorem 1.1.** Let  $\gamma \in (0, 1)$  and  $\delta \geq \frac{1}{2}$ . Then, for any  $\theta_0$  satisfying (1.4), there exists a weak solution of (1.2) in  $\mathcal{A}_T$  for all  $T > 0$ . Moreover, a weak solution is unique when  $\gamma = 1$ .

When  $\gamma = 1$ , we consider infinite energy solutions of (1.2). More precisely, we take a family of weights  $w_\beta = (1 + |x|^2)^{-\frac{\beta}{2}}$ ,  $0 < \beta < 1$ , and take initial data satisfying

$$\theta_0(x) \geq 0, \quad \theta_0 \in H^{\frac{1}{2}}(w_\beta dx) \cap L^\infty \quad (1.5)$$

where weighted Sobolev spaces are defined in Section 2. We note that  $\theta_0$  can decay (slowly) at infinity. For example, as long as  $\beta + 2\eta > 1$ ,  $|\theta_0(x)| \simeq |x|^{-\eta}$ ,  $\eta \geq 1/2$  is allowed to stay in  $L^2(w_\beta dx)$

$$\int_{|x| \geq 1} \frac{|x|^{2\eta}}{(1 + |x|^2)^{\frac{\beta}{2}}} dx < \infty.$$

But, we can still use the energy method to obtain a weak solution of (1.2). Let

$$\mathcal{B}_T = L^\infty \left(0, T; H^{\frac{1}{2}}(w_\beta dx)\right) \cap L^2 \left(0, T; H^1(w_\beta dx)\right).$$

**Theorem 1.2.** Let  $\gamma = 1$  and  $\delta \geq \frac{1}{2}$ . Then, for any  $\theta_0$  satisfying (1.5) with  $\|\theta_0\|_{L^\infty}$  being sufficiently small, there exists a unique weak solution of (1.2) in  $\mathcal{B}_T$  for all  $T > 0$ .

In Theorem 1.1 and Theorem 1.2, we have restrictions on the sign of initial data and the range of  $\delta$ . We can remove these conditions by looking for a solution of (1.2) in function spaces defined by the Fourier transform. Let

$$A^\alpha = \left\{ f \in L^1_{loc} : \|f\|_{A^\alpha} = \int_{\mathbb{R}} (1 + |\xi|^\alpha) |\hat{f}(\xi)| d\xi < \infty \right\}.$$

We also define

$$\mathcal{W}_T = L^\infty(0, T; W^{1,\infty}) \cap W^{1,\infty}(0, T; L^\infty) \cap L^1(0, T; W^{2,\infty}).$$

**Theorem 1.3.** *Let  $\gamma = 1$  and  $\delta \in \mathbb{R}$ . Then, for any  $\theta_0 \in A^1$  with*

$$\|\theta_0\|_{A^0} < \frac{\sqrt{\pi}}{\sqrt{2}(1 + |\delta|)}, \quad (1.6)$$

*there exists a unique weak solution of (1.2) verifying the following inequality for all  $T > 0$*

$$\theta \in \mathcal{W}_T, \quad \sup_{t \in [0, T]} \|\theta(t)\|_{A^1} + \left(1 - \frac{\sqrt{2}(1 + |\delta|)\|\theta_0\|_{A^0}}{\sqrt{\pi}}\right) \int_0^T \|\theta_x(t)\|_{A^1} dt \leq \|\theta_0\|_{A^1}.$$

We note that  $\theta_0 \in A^1$  can have infinite energy. For example, we take  $\hat{\theta}_0(\xi) = \frac{e^{-|\xi|}}{\sqrt{|\xi|}}$  for  $\xi \neq 0$ . Then,  $\theta_0 \in A^1$  but  $\theta_0 \notin L^2$ .

**1.2. The case  $\mathcal{N} = (1 - \partial_{xx})^{-\alpha}$  and  $\delta = 0$ .** In this case, (1.1) is changed to the equation

$$\theta_t + u\theta_x + \Lambda^\gamma \theta = 0, \quad u = (1 - \partial_{xx})^{-\alpha} \theta. \quad (1.7)$$

This equation is closely related to a generalized Proudman-Johnson equation [25, 27, 31]:

$$f_{txx} + f f_{xxx} + \delta f_x f_{xx} = \nu f_{xxxx}$$

which is derived from the 2D incompressible Navier-Stokes equations via the separation of space variables when  $\delta = 1$ . By taking  $w = f_{xx}$ ,

$$w_t + f w_x + \delta f_x w = \nu w_{xx}, \quad f = (\partial_{xx})^{-1} w.$$

The inviscid case with  $\delta = 2$  is equivalent to the Hunter-Saxton equation arising in the study of nematic liquid crystals [15]. The equation (1.7) is also considered as a model equation of the Lagrangian averaged Navier-Stokes equations [21] which are given by

$$\partial_t (1 - \sigma^2 \Delta) u + u \cdot \nabla (1 - \sigma^2 \Delta) u + (\nabla u)^T \cdot (1 - \sigma^2 \Delta) u = -\nabla p + \nu \Delta (1 - \sigma^2 \Delta) u, \quad \nabla \cdot u = 0$$

We first deal with (1.7) with initial data in  $L^2 \cap L^\infty$ . Let

$$\mathcal{C}_T = L^\infty(0, T; L^p) \cap L^2\left(0, T; H^{\frac{\gamma}{2}}\right) \quad \text{for all } p \in [2, \infty].$$

**Definition 1.2.** We say  $\theta$  is a weak solution of (1.7) if  $\theta \in \mathcal{C}_T$  and (1.7) holds in the following sense: for any test function  $\psi \in \mathcal{C}_c^\infty([0, T] \times \mathbb{R})$ ,

$$\int_0^T \int_{\mathbb{R}} [\theta \psi_t + u_x \theta \psi + u \theta \psi_x - \theta \Lambda^\gamma \psi] dx dt = \int_{\mathbb{R}} \theta_0(x) \psi(0, x) dx$$

holds for any  $0 < T < \infty$ .

**Theorem 1.4.** *Let  $\gamma \in (0, 2)$  and  $\alpha = \frac{1}{2} - \frac{\gamma}{4}$ . Then, for any  $\theta_0 \in L^2 \cap L^\infty$ , there exists a weak solution of (1.7) in  $\mathcal{C}_T$  for all  $T > 0$ .*

We note that  $\theta_0 \in L^2 \cap L^\infty$  is enough to construct a weak solution in Theorem 1.4, but we do not know whether it is unique or not.

When  $\gamma = 1$ , we consider weights  $w_\beta = (1 + |x|^2)^{-\frac{\beta}{2}}$  with  $0 < \beta < 1$ , and take initial data in  $H^1(w_\beta dx) \cap L^\infty$ . Let  $\alpha = \frac{1}{4}$  and

$$\mathcal{D}_T = L^\infty(0, T; H^1(w_\beta dx)) \cap L^2\left(0, T; H^{\frac{3}{2}}(w_\beta dx)\right).$$

**Theorem 1.5.** *Let  $\gamma = 1$  and  $\alpha = \frac{1}{4}$ . Then, for any  $\theta_0 \in H^1(w_\beta dx) \cap L^\infty$ , there exists a unique global weak solution of (1.7) in  $\mathcal{D}_T$  for all  $T > 0$ .*

Compared to Theorem 1.2, we do not assume that  $\|\theta_0\|_{L^\infty}$  is small to prove Theorem 1.5.

In Theorem 1.4 and Theorem 1.5, we have restrictions on the range of  $\alpha$ . Again, we can remove these conditions by looking for a solution of (1.7) in function spaces defined by the Fourier variables.

**Theorem 1.6.** *Let  $\gamma = 1$  and  $\alpha \geq 0$ . Then, for any  $\theta_0 \in A^1$  satisfying*

$$\|\theta_0\|_{A^0} < \frac{\sqrt{\pi}}{\sqrt{2}}, \quad (1.8)$$

*there exists a unique weak solution of (1.7) verifying the following inequality for all  $T > 0$*

$$\theta \in \mathcal{W}_T, \quad \sup_{t \in [0, T]} \|\theta(t)\|_{A^1} + \left(1 - \frac{\sqrt{2}\|\theta_0\|_{A^0}}{\sqrt{\pi}}\right) \int_0^T \|\theta_x(t)\|_{A^1} dt \leq \|\theta_0\|_{A^1}.$$

**Remark 1.** We note that Theorem 1.6 remains valid with straightforward changes in the spirit of Theorem 1.3 when  $\delta \neq 0$ .

## 2. PRELIMINARIES

All constants will be denoted by  $C$  that is a generic constant. In a series of inequalities, the value of  $C$  can vary with each inequality. For  $s \in \mathbb{R}$ ,  $H^s$  is a Hilbert space with

$$\|f\|_{H^s}^2 = \int_{\mathbb{R}} (1 + |\xi|^2)^s \left| \hat{f}(\xi) \right|^2 d\xi.$$

**2.1. Hilbert transform and fractional Laplacian.** The Hilbert transform is defined as

$$\mathcal{H}f(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f(y)}{x - y} dy.$$

The differential operator  $\Lambda^\gamma = (\sqrt{-\Delta})^\gamma$  is defined by the action of the following kernels [11]:

$$\Lambda^\gamma f(x) = c_\gamma \text{p.v.} \int_{\mathbb{R}} \frac{f(x) - f(y)}{|x - y|^{1+\gamma}} dy, \quad (2.1)$$

where  $c_\gamma > 0$  is a normalized constant. When  $\gamma = 1$ ,

$$\Lambda f(x) = \mathcal{H}f_x(x).$$

Moreover, we have the following identity:

$$\mathcal{H}(\theta_x (\mathcal{H}\theta_x)) = \frac{1}{2} \left[ (\Lambda\theta)^2 - (\theta_x)^2 \right]. \quad (2.2)$$

We also recall the following pointwise property of  $\Lambda^\alpha$ .

**Lemma 2.1.** [11] *Let  $0 \leq \alpha \leq 2$  and  $f \in \mathcal{S}$ . Then,*

$$\begin{aligned} f(x) \Lambda^\alpha f(x) &\geq \frac{1}{2} \Lambda^\alpha (f^2(x)), \\ f^2(x) \Lambda f(x) &\geq \frac{1}{3} \Lambda (f^3(x)) \quad \text{when } f \geq 0 \end{aligned}$$

**2.2. Minimum and Maximum Principles.** In Theorem 1.1 and 1.2, we assume  $\theta_0 > 0$ . To obtain global-in-time solutions, we need  $\theta(t, x) \geq 0$  for all time. We first assume that  $\theta(t, x) \in C^1([0, T] \times \mathbb{R})$  and  $x_t$  be a point such that  $m(t) = \theta(t, x_t)$ . If  $m(t) > 0$  for all time, nothing is left to prove. So, we check a point  $(t, x_t)$  where  $m(t) = 0$ . Since  $m(t)$  is a continuous Lipschitz function, it is differentiable at almost every  $t$  by Rademacher's theorem. From the definition of  $\Lambda^\gamma$ ,

$$\begin{aligned} \frac{d}{dt}m(t) &= -\delta\theta(t, x_t)\text{p.v.} \int_{\mathbb{R}} \frac{\theta(t, x_t) - \theta(t, y)}{|x_t - y|^{1+\gamma}} dy - \text{p.v.} \int_{\mathbb{R}} \frac{\theta(t, x_t) - \theta(t, y)}{|x_t - y|^{1+\gamma}} dy \\ &\geq \left[ -\delta\text{p.v.} \int_{\mathbb{R}} \frac{\theta(t, x_t) - \theta(t, y)}{|x_t - y|^{1+\gamma}} dy \right] m(t). \end{aligned}$$

Since the quantity in the bracket is nonnegative when  $\delta \geq 0$ , we have that  $m(t)$  is non-decreasing in time if  $\theta_0 > 0$  and thus  $\theta(t, x) \geq 0$  for all time. Similarly, maximum values of  $\theta(t, x)$  are non-increasing in time when  $\theta_0 > 0$  with  $\theta_0 \in L^\infty$ . For general initial data satisfying (1.4) and (1.5), we can use regularization method. For such a regularized problem with smooth solution  $\theta^\epsilon$ , the same argument works. Then, we construct  $\theta$  as the limit of  $\theta^\epsilon$ . As  $\theta$  will be also the pointwise limit of  $\theta^\epsilon$  almost everywhere, we conclude that  $\theta(t, x) \geq 0$ .

Since (1.7) is purely a dissipative transport equation, we immediately have that

$$\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}.$$

**2.3. The Wiener spaces  $A^\alpha$ .** The Wiener space is defined as

$$A^0 = \left\{ f \in L^1_{loc} : \hat{f}(\xi) \in L^1 \right\},$$

where  $\hat{f}$  denotes the Fourier transform of  $f$

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ix \cdot \xi} dx.$$

$A^0$  is a Banach space endowed with the norm

$$\|f\|_{A^0} = \|\hat{f}\|_{L^1}.$$

Furthermore, using Fubini's Theorem,  $A^0$  is a Banach algebra, *i.e.*

$$\|fg\|_{A^0} \leq \|f\|_{A^0} \|g\|_{A^0}.$$

Once we have defined  $A^0$ , we can define the full scale of homogeneous,  $\dot{A}^\alpha$ , and inhomogeneous,  $A^\alpha$ , Wiener spaces as

$$\begin{aligned} \dot{A}^\alpha &= \left\{ f \in L^1_{loc} : \|f\|_{\dot{A}^\alpha} = \int_{\mathbb{R}} |\xi|^\alpha |\hat{f}(\xi)| d\xi < \infty \right\}, \\ A^\alpha &= \left\{ f \in L^1_{loc} : \|f\|_{A^\alpha} = \int_{\mathbb{R}} (1 + |\xi|^\alpha) |\hat{f}(\xi)| d\xi < \infty \right\}. \end{aligned} \tag{2.3}$$

For these spaces, the following inequalities hold

$$\|f\|_{C(\mathbb{R})} \leq \|f\|_{A^0(\mathbb{R}^d)} \quad \forall f \in A^0(\mathbb{R}) \tag{2.4}$$

$$\|f\|_{\dot{A}^\alpha(\mathbb{R})} \leq \|f\|_{A^0(\mathbb{R}^d)}^{1-\theta} \|f\|_{\dot{A}^{\frac{\alpha}{\theta}}(\mathbb{R}^d)}^\theta \quad \forall 0 < \theta < 1, \alpha \geq 0, f \in A^0(\mathbb{R}) \cap \dot{A}^{\frac{\alpha}{\theta}}(\mathbb{R}). \tag{2.5}$$

As a consequence of (2.4), we obtain that if  $u \in A^0$  has infinite energy then

$$\limsup_{|x| \rightarrow \infty} |u(x)| + \liminf_{|x| \rightarrow \infty} |u(x)| < \infty.$$

**2.4. Commutator estimate.** In the proof of Theorem 1.1, we need to estimate a commutator term involving  $\Lambda^{\frac{1}{2}}$ . To do this, we first recall Hardy-Littlewood-Sobolev inequality in 1D. Let  $K_\alpha(x) = \frac{1}{|x|^\alpha}$  and  $T_\lambda f = K_\lambda * f$ . Then,

$$\|T_\lambda f\|_{L^q} \leq C \|f\|_{L^p}, \quad \frac{1}{q} + 1 = \frac{1}{p} + \lambda.$$

**Lemma 2.2.** For  $f \in L^{\frac{3}{2}}$ ,  $g \in L^{\frac{3}{2}}$  and  $\psi \in W^{1,\infty}$ ,

$$\left\| \left[ \Lambda^{\frac{1}{2}}, \psi \right] f - \left[ \Lambda^{\frac{1}{2}}, \psi \right] g \right\|_{L^6} \leq C \|\psi\|_{W^{1,\infty}} \|f - g\|_{L^{\frac{3}{2}}}.$$

*Proof.* By the definition of  $\Lambda^{\frac{1}{2}}$ , we have

$$\left( \left[ \Lambda^{\frac{1}{2}}, \psi \right] f - \left[ \Lambda^{\frac{1}{2}}, \psi \right] g \right)(x) = c_1 \text{p.v.} \int \frac{(\psi(y) - \psi(x))(f(y) - g(y))}{|x - y|^{\frac{3}{2}}} dy$$

and thus

$$\left| \left[ \Lambda^{\frac{1}{2}}, \psi \right] f - \left[ \Lambda^{\frac{1}{2}}, \psi \right] g \right|(x) \leq C \|\nabla \psi\|_{L^\infty} \int \frac{|f(y) - g(y)|}{|x - y|^{\frac{1}{2}}} dy. \quad (2.6)$$

Using Hardy-Littlewood-Sobolev inequality, we obtain that

$$\left\| \left[ \Lambda^{\frac{1}{2}}, \psi \right] f - \left[ \Lambda^{\frac{1}{2}}, \psi \right] g \right\|_{L^6} \leq C \|\nabla \psi\|_{L^\infty} \|f - g\|_{L^{\frac{3}{2}}} \quad (2.7)$$

which completes the proof.  $\square$

**2.5. Muckenhoupt weights.** We briefly introduce weighted spaces. A weight  $w$  is a positive and locally integrable function. A measurable function  $\theta$  on  $\mathbb{R}$  belongs to the weighted Lebesgue spaces  $L^p(wdx)$  with  $1 \leq p < \infty$  if and only if

$$\|\theta\|_{L^p(wdx)}^p = \int_{\mathbb{R}} |\theta(x)|^p w(x) dx < \infty.$$

An important class of weights is the Muckenhoupt class  $\mathcal{A}_p$  for  $1 < p < \infty$  [7, 24]. Let  $1 < p < \infty$ , we say that  $w \in \mathcal{A}_p$  if and only if there exists a constant  $C_{p,w} > 0$  such that

$$\sup_{r>0, x_0 \in \mathbb{R}} \left( \frac{1}{2r} \int_{[x_0-r, x_0+r]} w dx \right) \left( \frac{1}{2r} \int_{[x_0-r, x_0+r]} w^{\frac{1}{1-p}} dx \right)^{p-1} \leq C_{p,w}.$$

This class satisfies the following properties.

- (1) Calderón-Zygmund type operators are bound on  $L^p(wdx)$  when  $w \in \mathcal{A}_p$  and  $1 < p < \infty$  [29].
- (2) Let  $w \in \mathcal{A}_p$ . We define weighted Sobolev spaces as follows

$$\begin{aligned} f \in H^1(wdx) &\iff f \in L^2(wdx) \text{ and } f_x \in L^2(wdx), \\ f \in H^1(wdx) &\iff (1 - \partial_{xx})^{\frac{1}{2}} f \in L^2(wdx) \iff f \in L^2(wdx) \text{ and } \Lambda f \in L^2(wdx), \\ f \in H^{\frac{1}{2}}(wdx) &\iff (1 - \partial_{xx})^{\frac{1}{4}} f \in L^2(wdx) \iff f \in L^2(wdx) \text{ and } \Lambda^{\frac{1}{2}} f \in L^2(wdx). \end{aligned} \quad (2.8)$$

- (3) Gagliardo-Nirenberg type inequalities (see e.g [22])

$$\begin{aligned} \left\| \Lambda^{\frac{1}{2}} f \right\|_{L^2(wdx)} &\leq C \|f\|_{L^2(wdx)}^{\frac{1}{2}} \|\Lambda f\|_{L^2(wdx)}^{\frac{1}{2}}, \\ \|\theta\|_{L^4(wdx)} &\leq C \|\theta\|_{L^2(wdx)}^{\frac{1}{2}} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^{\frac{1}{2}}. \end{aligned} \quad (2.9)$$

This latter inequality can be proved for instance by using the weighted Sobolev embedding  $H^{\frac{1}{4}}(wdx) \hookrightarrow L^4(wdx)$ , and then by weighted interpolation one recover the second inequality in (2.9).

In this paper, we take weights  $w_\beta = (1 + |x|^2)^{-\frac{\beta}{2}}$ ,  $0 < \beta < 1$ , which belongs to the  $\mathcal{A}_p$  class of Muckenhoupt for all  $1 < p < \infty$ . These weights also satisfy the following properties. For the proofs, see [18].

**Lemma 2.3.** *Let  $w_\beta = (1 + |x|^2)^{-\frac{\beta}{2}}$ ,  $0 < \beta < 1$ .*

- (1) *For  $2 \leq p < \infty$  such that  $\beta(1 - p^{-1}) < 1/2$ , the commutator  $\frac{1}{w_\beta} [\Lambda^{\frac{1}{2}}, w_\beta]$  is bounded from  $L^p(w_\beta dx)$  to  $L^p(w_\beta dx)$ .*
- (2)  *$|\partial_x w_\beta(x)| \leq C w_\beta(x)$  and  $|\Lambda w_\beta(x)| \leq C w_\beta(x)$ , where  $C > 0$  depends only on  $\beta$ .*

Note that one can also derive commutator estimates for generalized Muckenhoupt weights of the type  $w_\beta = (1 + |x|^k)^{-\frac{\beta}{k}} \in \mathcal{A}_\infty$  where  $k$  is an even integer (see [17] for instance). However, our aim here is just to show the existence of global infinite energy solutions not to be optimal in the family of weights.

**2.6. Compactness.** Since we look for weak solutions, we use compactness arguments when we pass to the limit in weak formulations.

**Lemma 2.4.** [30] *Let  $X_0, X, X_1$  be reflexive Banach spaces such that*

$$X_0 \subset\subset X \subset X_1,$$

*where  $X_0$  is compactly embedded in  $X$ . Let  $T > 0$  be a finite number and let  $\alpha_0$  and  $\alpha_1$  be two finite numbers such that  $\alpha_i > 1$ . Then,  $Y = \{u \in L^{\alpha_0}(0, T; X_0), \partial_t u \in L^{\alpha_1}(0, T; X_1)\}$  is compactly embedded in  $L^{\alpha_0}(0, T; X)$ .*

**Lemma 2.5** ([9]). *Consider a sequence  $(\theta^\epsilon) \in C([0, T] \times B_R(0))$  that is uniformly bounded in  $L^\infty([0, T], W^{1, \infty}(B_R(0)))$ . Assume further that the weak derivative  $\frac{d\theta^\epsilon}{dt}$  is in  $L^\infty([0, T], L^\infty(B_R(0)))$  (not necessarily uniform) and is uniformly bounded in  $L^\infty([0, T], W_*^{-2, \infty}(B_R(0)))$ . Finally suppose that  $\theta_x^\epsilon \in C([0, T] \times B_R(0))$ . Then there exists a subsequence of  $(\theta^\epsilon)$  that converges strongly in  $L^\infty([0, T] \times B_R(0))$ .*

### 3. PROOF OF THEOREM 1.1

**3.1. A priori estimates.** We first obtain a priori bounds of the equation

$$\theta_t + (\mathcal{H}\theta)\theta_x + \delta\theta\Lambda\theta + \Lambda^\gamma\theta = 0, \quad (3.1)$$

We note that by the minimum principle applied to (3.1), we have  $\theta(t, x) \geq 0$  for all  $t \geq 0$ .

To obtain  $H^{\frac{1}{2}}$  bound of  $\theta$ , we begin with the  $L^2$  bound. We multiply (3.1) by  $\theta$  and integrate over  $\mathbb{R}$ . Then,

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2}^2 + \left\| \Lambda^{\frac{\gamma}{2}} \theta \right\|_{L^2}^2 = - \int [(\mathcal{H}\theta)\theta_x\theta] dx - \delta \int [\theta^2 \Lambda \theta] dx = \left( \frac{1}{2} - \delta \right) \int [\theta^2 \Lambda \theta] dx.$$

Since  $\theta \geq 0$ , we have

$$\int [\theta^2 \Lambda \theta] dx = \int \int \frac{(\theta(x) - \theta(y))^2}{|x - y|^2} \cdot \frac{\theta(x) + \theta(y)}{2} dx dy \geq 0$$

and thus

$$\|\theta(t)\|_{L^2}^2 + 2 \int_0^t \left\| \Lambda^{\frac{\gamma}{2}} \theta(s) \right\|_{L^2}^2 ds \leq \|\theta_0\|_{L^2}^2. \quad (3.2)$$

We next estimate  $\theta$  in  $\dot{H}^{\frac{1}{2}}$ . We multiply (3.1) by  $\Lambda\theta$  and integrate over  $\mathbb{R}$ :

$$\frac{1}{2} \frac{d}{dt} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2}^2 + \left\| \Lambda^{\frac{1+\gamma}{2}} \theta \right\|_{L^2}^2 = - \int [(\mathcal{H}\theta) \theta_x \Lambda\theta] dx - \delta \int [\theta (\Lambda\theta)^2] dx.$$

By (2.2), we have

$$- \int [(\mathcal{H}\theta) \theta_x \Lambda\theta] dx = \int [\theta \mathcal{H}(\theta_x (\mathcal{H}\theta_x))] dx = \frac{1}{2} \int [\theta ((\Lambda\theta)^2 - (\theta_x)^2)] dx,$$

and hence

$$\frac{1}{2} \frac{d}{dt} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2}^2 + \left\| \Lambda^{\frac{1+\gamma}{2}} \theta \right\|_{L^2}^2 = \left( \frac{1}{2} - \delta \right) \int [\theta (\Lambda\theta)^2] dx - \frac{1}{2} \int [\theta (\theta_x)^2] dx \leq 0,$$

where we use the sign conditions  $\theta \geq 0$  and  $\delta \geq \frac{1}{2}$ . This leads to the inequality

$$\left\| \Lambda^{\frac{1}{2}} \theta(t) \right\|_{L^2}^2 + 2 \int_0^t \left\| \Lambda^{\frac{1+\gamma}{2}} \theta(s) \right\|_{L^2}^2 ds \leq \left\| \Lambda^{\frac{1}{2}} \theta_0 \right\|_{L^2}^2. \quad (3.3)$$

By (3.2) and (3.3), we obtain that

$$\|\theta(t)\|_{H^{\frac{1}{2}}}^2 + 2 \int_0^t \left\| \Lambda^{\frac{\gamma}{2}} \theta(s) \right\|_{H^{\frac{1}{2}}}^2 ds \leq \|\theta_0\|_{H^{\frac{1}{2}}}^2. \quad (3.4)$$

We finally estimate  $\theta$  in  $L^1$ . Since  $\theta \geq 0$ ,

$$\frac{d}{dt} \|\theta\|_{L^1} = \frac{d}{dt} \int \theta dx = (1 - \delta) \int \theta \Lambda \theta dx \leq C \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2}^2$$

and thus we conclude that

$$\|\theta(t)\|_{L^1} \leq \|\theta_0\|_{L^1} + C \int_0^t \|\theta(s)\|_{\dot{H}^{\frac{1}{2}}}^2 ds \leq \|\theta_0\|_{L^1} + Ct \|\theta_0\|_{H^{\frac{1}{2}}}^2. \quad (3.5)$$

**3.2. Approximation and passing to limit.** We first regularize initial data as  $\theta_0^\epsilon = \rho_\epsilon * \theta_0$  where  $\rho_\epsilon$  is a standard mollifier that preserve the positivity of the regularized initial data. We then regularize the equation by putting the Laplacian with the coefficient  $\epsilon$ :

$$\theta_t^\epsilon + (\mathcal{H}\theta^\epsilon) \theta_x^\epsilon + \delta \theta^\epsilon \Lambda \theta^\epsilon + \Lambda^\gamma \theta^\epsilon = \epsilon \theta_{xx}^\epsilon. \quad (3.6)$$

For the proof of the existence of a global-in-time smooth solution, see [18] (Section 6). Moreover,  $(\theta^\epsilon)$  satisfies that

$$\|\theta^\epsilon(t)\|_{L^1} + \|\theta^\epsilon(t)\|_{H^{\frac{1}{2}}}^2 + 2 \int_0^t \left\| \Lambda^{\frac{\gamma}{2}} \theta^\epsilon(s) \right\|_{H^{\frac{1}{2}}}^2 ds + \epsilon \|\nabla \theta^\epsilon\|_{H^{\frac{1}{2}}}^2 \leq \|\theta_0\|_{L^1} + C(1+t) \|\theta_0\|_{H^{\frac{1}{2}}}^2.$$

Therefore,  $(\theta_\epsilon)$  is bounded in  $\mathcal{A}_T$  uniformly in  $\epsilon > 0$ . From this, we have uniform bounds

$$\mathcal{H}\theta^\epsilon \in L^4(0, T; L^4), \quad \theta^\epsilon \in L^2(0, T; L^4)$$

and hence

$$((\mathcal{H}\theta^\epsilon) \theta^\epsilon)_x \in L^{\frac{4}{3}}(0, T; H^{-1}).$$

Moreover,

$$\Lambda^\gamma \theta^\epsilon + \epsilon \theta_{xx}^\epsilon \in L^2(0, T; H^{-1}).$$



To estimate  $\theta^\epsilon \Lambda \theta^\epsilon$ , we use the duality argument. For any  $\chi \in L^2(0, T; H^2)$ ,

$$\begin{aligned}
|\langle \theta^\epsilon \Lambda \theta^\epsilon, \chi \rangle| &\leq \int |\widehat{\theta^\epsilon \Lambda \theta^\epsilon}(\xi) \widehat{\chi}(\xi)| d\xi \leq \int \int |\widehat{\theta^\epsilon}(\xi - \eta)| |\eta| |\widehat{\theta^\epsilon}(\eta)| |\widehat{\chi}(\xi)| d\eta d\xi \\
&\leq \int \int |\widehat{\theta^\epsilon}(\xi - \eta)| \left( |\eta|^{\frac{1}{2}} \left( |\xi - \eta|^{\frac{1}{2}} + |\xi|^{\frac{1}{2}} \right) \right) |\widehat{\theta^\epsilon}(\eta)| |\widehat{\chi}(\xi)| d\eta d\xi \\
&= \int \int |\xi - \eta|^{\frac{1}{2}} |\widehat{\theta^\epsilon}(\xi - \eta)| |\eta|^{\frac{1}{2}} |\widehat{\theta^\epsilon}(\eta)| |\widehat{\chi}(\xi)| d\eta d\xi \\
&+ \int \int |\widehat{\theta^\epsilon}(\xi - \eta)| |\eta|^{\frac{1}{2}} |\widehat{\theta^\epsilon}(\eta)| |\xi|^{\frac{1}{2}} |\widehat{\chi}(\xi)| d\eta d\xi \\
&\leq \left\| \Lambda^{\frac{1}{2}} \theta^\epsilon \right\|_{L^2}^2 \|\widehat{\chi}\|_{L^1} + \left\| \Lambda^{\frac{1}{2}} \theta^\epsilon \right\|_{L^2} \left\| \widehat{\theta} \right\|_{L^1} \left\| \Lambda^{\frac{1}{2}} \chi \right\|_{L^2} \leq C \left\| (1 + \Lambda)^{\frac{1}{2} + \frac{\gamma}{4}} \theta^\epsilon \right\|_{L^2}^2 \|\chi\|_{H^2}.
\end{aligned}$$

Since  $(1 + \Lambda)^{\frac{1}{2} + \frac{\gamma}{4}} \theta^\epsilon \in L^4(0, T; L^2)$  uniformly in  $\epsilon > 0$ , we have

$$\int |\langle \theta^\epsilon \Lambda \theta^\epsilon, \chi \rangle| dt \leq C \left\| (1 + \Lambda)^{\frac{1}{2} + \frac{\gamma}{4}} \theta^\epsilon \right\|_{L_T^4 L^2}^2 \|\chi\|_{L_T^2 H^2}.$$

This implies that  $\theta^\epsilon \Lambda \theta^\epsilon \in L^2(0, T; H^{-2})$ . So, we conclude that from the equation of  $\theta_t^\epsilon$

$$\theta_t^\epsilon \in L^{\frac{4}{3}}(0, T; H^{-2}).$$

We now extract a subsequence of  $(\theta^\epsilon)$ , using the same index  $\epsilon$  for simplicity, and a function  $\theta \in \mathcal{A}_T$  such that

$$\begin{aligned}
\theta^\epsilon &\overset{*}{\rightharpoonup} \theta \quad \text{in } L^\infty\left(0, T; L^p \cap H^{\frac{1}{2}}\right) \quad \text{for all } p \in (1, \infty), \\
\theta^\epsilon &\rightharpoonup \theta \quad \text{in } L^2\left(0, T; H^{\frac{\gamma+1}{2}}\right), \\
\theta^\epsilon &\rightarrow \theta \quad \text{in } L^2\left(0, T; H^{\frac{1}{2}}\right), \\
\theta^\epsilon &\rightarrow \theta \quad \text{in } L^2\left(0, T; L_{\text{loc}}^p\right) \quad \text{for all } p \in (1, \infty)
\end{aligned} \tag{3.7}$$

where we use Lemma 2.4 for the strong convergence.

We now multiply (3.6) by a test function  $\psi \in C_c^\infty([0, T] \times \mathbb{R})$  and integrate over  $\mathbb{R}$ . Then,

$$\int_0^T \int [\theta^\epsilon \psi_t + (\mathcal{H}\theta^\epsilon) \theta^\epsilon \psi_x + (1 - \delta) \Lambda \theta^\epsilon \theta^\epsilon \psi - \theta^\epsilon \Lambda \psi + \epsilon \theta^\epsilon \psi_{xx}] dx dt = \int \theta_0^\epsilon(x) \psi(0, x) dx$$

which can be rewritten as

$$\begin{aligned}
&\int_0^T \int \left[ \theta^\epsilon \psi_t + \underbrace{(\mathcal{H}\theta^\epsilon) \theta^\epsilon \psi_x}_{\text{I}} - \theta^\epsilon \Lambda \psi + \epsilon \theta^\epsilon \psi_{xx} \right] dx dt - \int \theta_0^\epsilon(x) \psi(0, x) dx \\
&= -(1 - \delta) \int_0^T \int \underbrace{\Lambda^{\frac{1}{2}} \theta^\epsilon \left[ \Lambda^{\frac{1}{2}}, \psi \right] \theta^\epsilon}_{\text{II}} dx dt - (1 - \delta) \int_0^T \int \underbrace{\left| \Lambda^{\frac{1}{2}} \theta^\epsilon \right|^2 \psi}_{\text{III}} dx dt.
\end{aligned} \tag{3.8}$$

By Lemma 2.4 with

$$X_0 = L^2\left(0, T; H^{\frac{1}{2}}\right), \quad X = L^2\left(0, T; L_{\text{loc}}^2\right), \quad X_1 = L^2\left(0, T; H^{-2}\right),$$

we can pass to the limit to I. Moreover, since

$$\left[ \Lambda^{\frac{1}{2}}, \psi \right] \theta^\epsilon \rightarrow \left[ \Lambda^{\frac{1}{2}}, \psi \right] \theta$$

strongly in  $L^2(0, T; L^6)$  by Lemma 2.2 and  $\Lambda^{\frac{1}{2}}\theta^\epsilon$  converges weakly in  $L^2(0, T; L^2)$  by (3.7), we can pass to the limit to II. Lastly, Lemma 2.4 with

$$X_0 = L^2\left(0, T; H^{\frac{1+\gamma}{2}}\right), \quad X = L^2\left(0, T; H_{\text{loc}}^{\frac{1}{2}}\right), \quad X_1 = L^2(0, T; H^{-2}),$$

allows to pass to the limit to III. Combining all the limits together, we obtain that

$$\int_0^T \int [\theta \psi_t + (\mathcal{H}\theta) \theta \psi_x + (1 - \delta) \Lambda \theta \theta^\epsilon \psi] dx dt = \int \theta_0(x) \psi(0, x) dx. \quad (3.9)$$

**3.3. Uniqueness when  $\gamma = 1$ .** To show the uniqueness of a weak solution, let  $\theta = \theta_1 - \theta_2$ . Then,  $\theta$  satisfies the following equation:

$$\theta_t + \Lambda \theta = -(\mathcal{H}\theta) \theta_{1x} - (\mathcal{H}\theta_2) \theta_x - \delta \theta \Lambda \theta_1 - \delta \theta_2 \Lambda \theta, \quad \theta(0, x) = 0. \quad (3.10)$$

We multiply  $\theta$  to (3.10) and integrate over  $\mathbb{R}$ . Then,

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2}^2 + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2}^2 = \int [ -(\mathcal{H}\theta) \theta_{1x} - (\mathcal{H}\theta_2) \theta_x - \delta \theta \Lambda \theta_1 - \delta \theta_2 \Lambda \theta ] \theta dx.$$

The first three terms in the right-hand side are easily bounded by

$$C \|\theta_{1x}\|_{L^2} \|\theta\|_{L^4}^2 + C \|\theta_{2x}\|_{L^2} \|\theta\|_{L^4}^2.$$

Moreover, the last term is bounded by using Lemma 2.1

$$-\delta \int \theta_2 \theta \Lambda \theta dx \leq -\frac{\delta}{2} \int \theta_2 \Lambda \theta^2 dx = -\frac{\delta}{2} \int \theta^2 \Lambda \theta_2 dx \leq C \|\theta_{2x}\|_{L^2} \|\theta\|_{L^4}^2.$$

Hence we derive that

$$\begin{aligned} \frac{d}{dt} \|\theta\|_{L^2}^2 + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2}^2 &\leq C \|\theta_{1x}\|_{L^2} \|\theta\|_{L^4}^2 + C \|\theta_{2x}\|_{L^2} \|\theta\|_{L^4}^2 \\ &\leq C \left( \|\theta_{1x}\|_{L^2}^2 + \|\theta_{2x}\|_{L^2}^2 \right) \|\theta\|_{L^2}^2 + \frac{1}{2} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2}^2. \end{aligned}$$

Since

$$\theta_{1x} \in L^2(0, T; L^2), \quad \theta_{2x} \in L^2(0, T; L^2)$$

when  $\gamma = 1$ , we conclude that  $\theta = 0$  in  $L^2$  and thus a weak solution is unique. This completes the proof of Theorem 1.1.

#### 4. PROOF OF THEOREM 1.2

**4.1. A priori estimates.** We consider the equation

$$\theta_t + (\mathcal{H}\theta) \theta_x + \delta \theta \Lambda \theta + \Lambda \theta = 0. \quad (4.1)$$

Since (4.1) satisfies the minimum and maximum principles, we have

$$\theta(t, x) \geq 0, \quad \|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}.$$

We begin with the  $L^2(w_\beta dx)$  bound. For notational simplicity, we suppress the dependence of  $\beta$ . We multiply (4.1) by  $\theta w$  and integrate in  $x$ . Then,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int \|\theta\|_{L^2(w_\beta dx)}^2 + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(w_\beta dx)}^2 &= - \int (\mathcal{H}\theta) \theta_x \theta w dx - \delta \int \theta (\Lambda \theta) \theta w dx - \int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx \\ &= -\frac{1}{2} \int (\mathcal{H}\theta) (\theta^2)_x w dx - \delta \int \theta^2 (\Lambda \theta) w dx - \int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx \\ &= \left( \frac{1}{2} - \delta \right) \int \theta^2 (\Lambda \theta) w dx + \frac{1}{2} \int (\mathcal{H}\theta) \theta^2 w_x dx - \int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx. \end{aligned}$$

Since  $\frac{1}{2} - \delta \leq 0$  and  $\theta \geq 0$ , we use Lemma 2.1 to have

$$\begin{aligned} \left(\frac{1}{2} - \delta\right) \int \theta^2 (\Lambda\theta) w dx &\leq -\frac{1}{3} \left(\frac{1}{2} - \delta\right) \int \Lambda (\theta^3) w dx = -\frac{1}{3} \left(\frac{1}{2} - \delta\right) \int \theta^3 \Lambda w dx \\ &\leq C \|\theta\|_{L^\infty} \int \theta^2 w dx \leq C \|\theta_0\|_{L^\infty} \|\theta\|_{L^2(wdx)}^2, \end{aligned}$$

where we use Lemma 2.3 to bound  $\Lambda w$  by  $w$ . Moreover, by the  $L^2(wdx)$  boundedness of the Hilbert transform, we also have

$$\frac{1}{2} \int (\mathcal{H}\theta) \theta^2 w dx \leq C \|\theta\|_{L^\infty} \int |\theta| |\mathcal{H}\theta| w dx \leq C \|\theta_0\|_{L^\infty} \|\theta\|_{L^2(wdx)}^2.$$

We finally estimate the commutator term. By Lemma 2.3,

$$\begin{aligned} \left| \int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx \right| &\leq \int \left| \left( \Lambda^{\frac{1}{2}} \theta \right) w \frac{1}{w} \left[ \Lambda^{\frac{1}{2}}, w \right] \theta \right| dx \leq C \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)} \left\| \frac{1}{w} \left[ \Lambda^{\frac{1}{2}}, w \right] \theta \right\|_{L^2(wdx)} \\ &\leq C \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)} \|\theta\|_{L^2(wdx)} \leq \frac{1}{2} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 + C \|\theta\|_{L^2(wdx)}^2. \end{aligned}$$

Collecting all terms together, we obtain that

$$\frac{d}{dt} \|\theta\|_{L^2(wdx)}^2 + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 \leq C(1 + \|\theta_0\|_{L^\infty}) \|\theta\|_{L^2(wdx)}^2. \quad (4.2)$$

We next multiply (4.1) by  $\Lambda^{\frac{1}{2}} (w \Lambda^{\frac{1}{2}} \theta)$  and integrate in  $x$ . Then,

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 + \|\Lambda\theta\|_{L^2(wdx)} \\ &= - \int (\mathcal{H}\theta) \theta_x \Lambda^{\frac{1}{2}} (w \Lambda^{\frac{1}{2}} \theta) dx - \delta \int \theta (\Lambda\theta) \Lambda^{\frac{1}{2}} (w \Lambda^{\frac{1}{2}} \theta) dx - \int \Lambda\theta \left[ \Lambda^{\frac{1}{2}}, w \right] \Lambda^{\frac{1}{2}} \theta dx \\ &= \text{I} + \text{II} + \text{III}. \end{aligned}$$

We note that since  $\delta > 0$  and  $\theta \geq 0$

$$\text{II} = -\delta \int \theta |\Lambda\theta|^2 w dx - \delta \int \theta \Lambda\theta \left[ \Lambda^{\frac{1}{2}}, w \right] \Lambda^{\frac{1}{2}} \theta dx \leq -\delta \int \theta \Lambda\theta \left[ \Lambda^{\frac{1}{2}}, w \right] \Lambda^{\frac{1}{2}} \theta dx \quad (4.3)$$

and thus we only need to estimate I and III and the right-hand side of (4.3). (There is an extra term  $\theta$  in the right-hand side of (4.3) but we can take the  $L^\infty$  norm to  $\theta$  and it does not affect the proof.) These bounds are obtained in [18]:

$$\begin{aligned} \text{I} + \text{III} - \delta \int \Lambda\theta \left[ \Lambda^{\frac{1}{2}}, w \right] \Lambda^{\frac{1}{2}} \theta dx &\leq C (\|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^4) \|\theta\|_{L^2(wdx)}^2 + C \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 \\ &\quad + C \|\theta_0\|_{L^\infty} \|\Lambda\theta\|_{L^2(wdx)}^2 + \frac{1}{2} \|\Lambda\theta\|_{L^2(wdx)}^2. \end{aligned}$$

Hence we have that

$$\begin{aligned} &\frac{d}{dt} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 + \|\Lambda\theta\|_{L^2(wdx)}^2 \\ &\leq C (\|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^4) \|\theta\|_{L^2(wdx)}^2 + C \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 + C \|\theta_0\|_{L^\infty} \|\Lambda\theta\|_{L^2(wdx)}^2 \end{aligned} \quad (4.4)$$

By (4.2) and (4.4),

$$\begin{aligned} \frac{d}{dt} \|\theta\|_{H^{\frac{1}{2}}(wdx)}^2 + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 + \|\Lambda\theta\|_{L^2(wdx)}^2 &\leq C (\|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^4) \|\theta\|_{H^{\frac{1}{2}}(wdx)}^2 \\ &\quad + C \|\theta_0\|_{L^\infty} \|\Lambda\theta\|_{L^2(wdx)}^2. \end{aligned}$$

If  $\|\theta_0\|_{L^\infty}$  is sufficiently small,

$$\frac{d}{dt} \|\theta\|_{H^{\frac{1}{2}}(wdx)}^2 + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{H^{\frac{1}{2}}(wdx)}^2 \leq C (\|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^4) \|\theta\|_{H^{\frac{1}{2}}(wdx)}^2$$

and hence we derive the following inequality

$$\|\theta(t)\|_{H^{\frac{1}{2}}(wdx)}^2 + \int_0^t \left\| \Lambda^{\frac{1}{2}} \theta(s) \right\|_{H^{\frac{1}{2}}(wdx)}^2 ds \leq \|\theta_0\|_{H^{\frac{1}{2}}(wdx)}^2 \exp(C (\|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^4) t). \quad (4.5)$$

**4.2. Approximation and passing to limit.** To show the existence of a weak solution in  $\mathcal{D}_T$ , we first approximate the initial data  $\theta_0$ . Let  $\chi$  be a smooth positive function such that  $\chi(x) = 1$  for  $|x| \leq 1$  and  $\chi(x) = 0$  for  $|x| \geq 2$ . Let  $\chi_R(x) = \chi(x/N)$ ,  $N \in \mathbb{N}$ , and consider truncated initial data  $\theta_0^N(x) = \theta_0(x)\chi_N(x)$ . Then, a direct computation shows that

$$\lim_{N \rightarrow \infty} \|\theta_0^N - \theta_0\|_{H^{\frac{1}{2}}(wdx)} = 0.$$

Moreover, this truncation does not alter the non-negativity and does not increase the  $L^\infty$  norm. So, if  $\|\theta_0\|_{L^\infty}$  is sufficiently small, there is a global-in-time solution of

$$\partial_t \theta^N + \mathcal{H} \theta^N \partial_x \theta^N + \delta \theta^N \Lambda \theta^N + \Lambda \theta^N = 0, \quad \theta^N(0, x) = \theta_0^N(x). \quad (4.6)$$

From the a priori estimates, the sequence  $(\theta^N)$  is bounded in

$$L^\infty([0, T], H^{\frac{1}{2}}(wdx)) \cap L^2([0, T], H^1(wdx))$$

uniformly with respect to  $N$ . We now take a test function  $\psi \in \mathcal{C}_c^\infty([0, T] \times \mathbb{R})$ . Then,  $\psi \theta^N$  is bounded in  $L^2([0, T], H^1)$ . Moreover, since  $\theta^N \in L^\infty([0, T] \times \mathbb{R})$

$$\begin{aligned} & \psi (\mathcal{H} \theta^N \partial_x \theta^N + \delta \theta^N \Lambda \theta^N + \Lambda \theta^N) \\ &= (\psi \mathcal{H} \theta^N \theta^N)_x - \psi_x \mathcal{H} \theta^N \theta^N + (\delta - 1) \psi \theta^N \Lambda \theta^N + \psi \Lambda \theta^N \in L^2([0, T], H^{-1}). \end{aligned}$$

By Lemma 2.4, we can pass to the limit to the weak formulation,

$$\int_0^T \int [\theta^N \psi_t + (\mathcal{H} \theta^N) \theta^N \psi_x + (1 - \delta) \Lambda \theta^N \theta^N \psi - \theta^N \Lambda \psi] dx dt = \int \theta_0^N(x) \psi(0, x) dx,$$

to obtain a weak solution  $\theta$  which is also in

$$L^\infty([0, T], H^{\frac{1}{2}}(wdx)) \cap L^2([0, T], H^1(wdx)).$$

**4.3. Uniqueness.** To show the uniqueness of a weak solution, we consider the equation of  $\theta = \theta_1 - \theta_2$  given by

$$\theta_t + \Lambda \theta = -(\mathcal{H} \theta) \theta_{1x} - (\mathcal{H} \theta_2) \theta_x - \delta \theta \Lambda \theta_1 - \delta \theta_2 \Lambda \theta, \quad \theta(0, x) = 0. \quad (4.7)$$

We multiply  $w\theta$  to (4.7) and integrate over  $\mathbb{R}$ . Then,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2(wdx)}^2 + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 &= \int [-(\mathcal{H} \theta) \theta_{1x} - (\mathcal{H} \theta_2) \theta_x - \delta \theta \Lambda \theta_1 - \delta \theta_2 \Lambda \theta] \theta w dx \\ &\quad - \int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx. \end{aligned}$$

As before, the last term is bounded by

$$\int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx \leq \frac{1}{2} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 + C \|\theta\|_{L^2(wdx)}^2.$$

The first three terms in the right-hand side are easily bounded by

$$C \left( \|\theta_{1x}\|_{L^2(wdx)} + \|\theta_{2x}\|_{L^2(wdx)} + \|\theta_2\|_{L^2(wdx)} \right) \|\theta\|_{L^4(wdx)}^2.$$

Moreover, since  $\delta > 0$ ,  $\theta_2 \geq 0$  and  $w \geq 0$ , the fourth term is bounded by using Lemma 2.1

$$\begin{aligned} -\delta \int \theta_2 \theta \Lambda \theta w dx &\leq -\frac{\delta}{2} \int \theta_2 w \Lambda \theta^2 dx = -\frac{\delta}{2} \int \theta^2 \Lambda (\theta_2 w) dx = \frac{\delta}{2} \int \mathcal{H}(\theta^2) (\theta_2 w)_x dx \\ &\leq C \left( \|\theta_{2x}\|_{L^2(wdx)} + \|\theta_2\|_{L^2(wdx)} \right) \|\theta\|_{L^4(wdx)}^2. \end{aligned}$$

Hence we obtain that

$$\begin{aligned} &\frac{d}{dt} \|\theta\|_{L^2(wdx)}^2 + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 \\ &\leq C \left( \|\theta_{1x}\|_{L^2(wdx)} + \|\theta_{2x}\|_{L^2(wdx)} + \|\theta_2\|_{L^2(wdx)} \right) \|\theta\|_{L^4(wdx)}^2 + C \|\theta\|_{L^2(wdx)}^2 \\ &\leq C \left( 1 + \|\theta_{1x}\|_{L^2(wdx)}^2 + \|\theta_{2x}\|_{L^2(wdx)}^2 + \|\theta_2\|_{L^2(wdx)}^2 \right) \|\theta\|_{L^2(wdx)}^2 + \frac{1}{2} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2, \end{aligned}$$

where we use (2.9) to obtain the last inequality. Since

$$\theta_{1x} \in L^2(0, T; L^2(wdx)), \quad \theta_{2x} \in L^2(0, T; H^1(wdx)),$$

we conclude that  $\theta = 0$  in  $L^2(wdx)$  and thus a weak solution is unique. This completes the proof of Theorem 1.2.

## 5. PROOF OF THEOREM 1.3

5.1. **A priori estimates.** Taking the Fourier transform of (1.2), we have that

$$\begin{aligned} \partial_t |\hat{\theta}(\xi)| &= \frac{\bar{\hat{\theta}}(\xi) \partial_t \hat{\theta}(\xi) + \hat{\theta}(\xi) \partial_t \bar{\hat{\theta}}(\xi)}{2|\hat{\theta}(\xi)|} = \frac{\operatorname{Re} \left( \bar{\hat{\theta}}(\xi) \partial_t \hat{\theta}(\xi) \right)}{|\hat{\theta}(\xi)|} \\ &= -\operatorname{Re} \left[ \int_{\mathbb{R}} \frac{-i\zeta}{|\zeta|} \hat{\theta}(\zeta) i(\xi - \zeta) \hat{\theta}(\xi - \zeta) - \delta \hat{\theta}(\zeta) |\xi - \zeta| \hat{\theta}(\xi - \zeta) d\zeta \frac{\bar{\hat{\theta}}(\xi)}{|\hat{\theta}(\xi)|} \right] \frac{1}{\sqrt{2\pi}} - |\xi| |\hat{\theta}|. \end{aligned}$$

Consequently,

$$\begin{aligned} \frac{d}{dt} \|\theta\|_{A^0} &\leq (1 + |\delta|) \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{\theta}(\zeta)| |\xi - \zeta| |\hat{\theta}(\xi - \zeta)| d\zeta d\xi \frac{1}{\sqrt{2\pi}} - \|\theta\|_{A^1} \\ &\leq (1 + |\delta|) \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{\theta}(\zeta)| |\xi - \zeta| |\hat{\theta}(\xi - \zeta)| d\xi d\zeta \frac{1}{\sqrt{2\pi}} - \|\theta\|_{A^1} \leq \left( \frac{(1 + |\delta|) \|\theta\|_{A^0}}{\sqrt{2\pi}} - 1 \right) \|\theta\|_{A^1}. \end{aligned}$$

Thus, if  $\theta_0$  satisfies the condition (1.6), we have

$$\|\theta(t)\|_{A^0} + \left( 1 - \frac{(1 + |\delta|) \|\theta_0\|_{A^0}}{\sqrt{2\pi}} \right) \int_0^\infty \|\theta(s)\|_{A^1} ds \leq \|\theta_0\|_{A^0}. \quad (5.1)$$

Similarly,

$$\begin{aligned} \partial_t |\xi \hat{\theta}| &= -\operatorname{Re} \left[ \int_{\mathbb{R}} \frac{-i\zeta}{|\zeta|} \hat{\theta}(\zeta) (i(\xi - \zeta))^2 \hat{\theta}(\xi - \zeta) + |\zeta| \hat{\theta}(\zeta) i(\xi - \zeta) \hat{\theta}(\xi - \zeta) d\zeta \right. \\ &\quad \left. - \delta \int_{\mathbb{R}} i\zeta \hat{\theta}(\zeta) |\xi - \zeta| \hat{\theta}(\xi - \zeta) + \hat{\theta}(\zeta) i(\xi - \zeta) |\xi - \zeta| \hat{\theta}(\xi - \zeta) d\zeta \frac{i\xi \bar{\hat{\theta}}(\xi)}{|\xi| |\hat{\theta}(\xi)|} \right] \frac{1}{\sqrt{2\pi}} - |\xi|^2 |\hat{\theta}|. \end{aligned}$$

Thus, using (2.5), we have that

$$\frac{d}{dt} \|\theta\|_{A^1} \leq (\|\theta\|_{A^0} \|\theta\|_{A^2} + \|\theta\|_{A^1}^2) \frac{1 + |\delta|}{\sqrt{2\pi}} - \|\theta\|_{A^2} \leq \left( \frac{2(1 + |\delta|) \|\theta\|_{A^0}}{\sqrt{2\pi}} - 1 \right) \|\theta\|_{A^2}.$$

As a consequence, we obtain that, if  $\theta_0$  satisfies the condition (1.6), we also have

$$\|\theta(t)\|_{A^1} + \left( 1 - \frac{2(1 + |\delta|) \|\theta_0\|_{A^0}}{\sqrt{2\pi}} \right) \int_0^\infty \|\theta(s)\|_{A^2} ds \leq \|\theta_0\|_{A^1}. \quad (5.2)$$

By (5.1) and (5.2), we conclude that

$$\|\theta(t)\|_{A^1} + \left(1 - \frac{2(1 + |\delta|)\|\theta_0\|_{A^0}}{\sqrt{2\pi}}\right) \int_0^t \|\theta_x(s)\|_{A^1} ds \leq \|\theta_0\|_{A^1} \quad (5.3)$$

for all  $t \geq 0$ .

**5.2. Approximation and passing to the limit.** Define  $e^{\epsilon \partial_x^2}$  the heat semigroup, *i.e.*

$$e^{\epsilon \partial_x^2} f(x) = e^{-\epsilon \xi^2} \hat{f}(\xi),$$

and  $g_\epsilon(x) = e^{-\epsilon x^2}$ . Note that

$$\hat{g}_\epsilon(\xi) = \frac{1}{\sqrt{2\epsilon}} e^{-\frac{\xi^2}{4\epsilon}}, \quad \|\hat{g}_\epsilon\|_{L^1} = \sqrt{2\pi}.$$

Given  $\theta_0(x) \in A^1$ , we consider  $\theta_0^\epsilon(x) = g_\epsilon(x) e^{\epsilon \partial_x^2} \theta_0(x)$ . As  $\theta_0(x)$  is a bounded function, we have that  $\theta_0^\epsilon$  is infinitely smooth and has finite total mass:

$$\|\theta_0^\epsilon\|_{L^1} \leq \|g_\epsilon\|_{L^1} \|e^{\epsilon \partial_x^2} \theta_0\|_{L^\infty} \leq \sqrt{\frac{\pi}{\epsilon}} \|\theta_0\|_{L^\infty}.$$

Furthermore, using Young's inequality and the definition of  $g_\epsilon$ ,

$$\|\theta_0^\epsilon\|_{A^0} = \frac{1}{\sqrt{2\pi}} \|\hat{g}_\epsilon * (e^{-\epsilon \xi^2} \hat{\theta}_0)\|_{L^1} \leq \|e^{-\epsilon \xi^2} \hat{\theta}_0\|_{L^1} \leq \|\theta_0\|_{A^0}.$$

Similarly,

$$\|\theta_0^\epsilon\|_{A^1} = \|\partial_x \theta_0^\epsilon\|_{A^0} \leq \|\theta_0\|_{A^1} + \|\partial_x g_\epsilon e^{\epsilon \partial_x^2} \theta_0\|_{A^0} = \|\theta_0\|_{A^1} + c\epsilon \|\theta_0\|_{A^0}.$$

Now we define the approximated problems

$$\theta_t^\epsilon + (\mathcal{H}\theta^\epsilon) \theta^\epsilon + \delta \theta^\epsilon \Lambda \theta^\epsilon + \Lambda \theta^\epsilon = \epsilon \partial_x^2 \theta^\epsilon, \quad (5.4)$$

with finite energy approximated initial data  $\theta_0^\epsilon$ . These problems have unique smooth solutions denoted by  $\theta^\epsilon$ . Moreover,  $(\theta^\epsilon)$  satisfies a uniform bound

$$\|\theta^\epsilon(t)\|_{A^1} + \left(1 - \frac{2(1 + |\delta|)\|\theta_0\|_{A^0}}{\sqrt{2\pi}}\right) \int_0^t \|\theta_x^\epsilon(s)\|_{A^1} ds + \epsilon \int_0^t \|\theta_x^\epsilon(s)\|_{A^2} ds \leq \|\theta_0\|_{A^1}.$$

uniformly in  $\epsilon$ . Thus, the a priori estimates lead to the following uniform-in- $\epsilon$  bounds

$$\begin{aligned} \sup_{t \in [0, \infty)} \|\theta^\epsilon(t)\|_{C^0(\mathbb{R})} &\leq \sup_{t \in [0, \infty)} \|\theta^\epsilon(t)\|_{A^0} \leq \|\theta_0\|_{A^0} < \frac{\sqrt{2\pi}}{1 + |\delta|}, \\ \sup_{t \in [0, \infty)} \|\theta^\epsilon(t)\|_{\dot{C}^1(\mathbb{R})} &\leq \sup_{t \in [0, \infty)} \|\theta^\epsilon(t)\|_{A^1} < \|\theta_0\|_{A^1} + c\epsilon \|\theta_0\|_{A^0}, \\ \sup_{t \in [0, \infty)} \|\Lambda \theta^\epsilon(t)\|_{C^0(\mathbb{R})} &< \|\theta_0\|_{A^1} + c\epsilon \|\theta_0\|_{A^0}. \end{aligned} \quad (5.5)$$

Moreover, from the equation (5.4) we also obtain uniform bounds

$$\begin{aligned} \sup_{t \in [0, \infty)} \|\partial_t \theta^\epsilon(t)\|_{C^0(\mathbb{R})} + \sup_{t \in [0, \infty)} \|\partial_t H \theta^\epsilon(t)\|_{C^0(\mathbb{R})} &\leq 2 \sup_{t \in [0, \infty)} \|\partial_t \theta^\epsilon(t)\|_{A^0(\mathbb{R})} \leq F_1(\|\theta_0\|_{A^1}, \delta), \\ \|H \theta^\epsilon\|_{C^1([0, \infty) \times \mathbb{R})} + \|\theta^\epsilon\|_{C^1([0, \infty) \times \mathbb{R})} &\leq F_2(\|\theta_0\|_{A^1}, \delta) \end{aligned} \quad (5.6)$$

where  $F_1$  and  $F_2$  only depend on the quantities in the right-hand side of (5.5). Due to Banach-Alaoglu Theorem, there exists a subsequence (denoted by  $\epsilon$ ) and a limit function  $\theta \in W^{1, \infty}([0, \infty) \times \mathbb{R})$  such that

$$\theta^\epsilon \xrightarrow{*} \theta \quad \text{in } L^\infty(0, T; W^{1, \infty}), \quad \Lambda \theta^\epsilon \xrightarrow{*} \Lambda \theta \quad \text{in } L^\infty(0, T; L^\infty)$$

for all  $T > 0$ . Using Lemma 2.5, we have the following strong convergence

$$\lim_{\epsilon \rightarrow 0} \|\theta^\epsilon - \theta\|_{L^\infty(K)} + \|H \theta^\epsilon - H \theta\|_{L^\infty(K)} = 0$$

for  $K = [0, T] \times [-R, R]$  for any  $R > 0$ . Since  $L^\infty(K) \subset L^2(K)$ , we can pass to the limit to the weak formulation

$$\int_0^T \int [\theta^\epsilon \psi_t + (\mathcal{H}\theta^\epsilon) \theta^\epsilon \psi_x + (1 - \delta)\Lambda\theta^\epsilon \theta^\epsilon \psi - \theta^\epsilon \Lambda\psi + \epsilon\theta^\epsilon \psi_{xx}] dx dt = \int \theta_0^\epsilon(x) \psi(0, x) dx$$

to obtain a weak solution  $\theta$  satisfying

$$\theta \in \mathcal{W}_T, \quad \sup_{t \in [0, T]} \|\theta(t)\|_{A^1} + \left(1 - \frac{\sqrt{2}\|\theta_0\|_{A^0}}{\sqrt{\pi}}\right) \int_0^T \|\theta_x(t)\|_{A^1} dt \leq \|\theta_0\|_{A^1}$$

for all  $T > 0$ .

**5.3. Uniqueness.** To show the uniqueness of a weak solution, we consider the equation of  $\theta = \theta_1 - \theta_2$  given by

$$\theta_t + \Lambda\theta = -(\mathcal{H}\theta)\theta_{1x} - (\mathcal{H}\theta_2)\theta_x - \delta\theta\Lambda\theta_1 - \delta\theta_2\Lambda\theta, \quad \theta(0, x) = 0. \quad (5.7)$$

Taking the Fourier transform of (5.7) and multiply by  $\frac{\hat{\theta}}{|\theta|}$ , we have

$$\frac{d}{dt} \|\theta\|_{A^0} + \|\theta\|_{A^1} \leq \frac{1 + |\delta|}{\sqrt{2\pi}} \|\theta_1\|_{A^1} \|\theta\|_{A^0} + \frac{1 + |\delta|}{\sqrt{2\pi}} \|\theta_2\|_{A^0} \|\theta\|_{A^1}. \quad (5.8)$$

Since

$$\frac{(1 + |\delta|) \|\theta_2\|_{A^0}}{\sqrt{2\pi}} < 1, \quad \|\theta_1(t)\|_{A^1} \in L^1([0, \infty)),$$

we have  $\theta(t, x) = 0$  in  $A^0$  for all time. This implies that a weak solution is unique.

## 6. PROOF OF THEOREM 1.4

**6.1. A priori estimates.** We consider the equation

$$\theta_t + u\theta_x + \Lambda^\gamma\theta = 0, \quad u = (1 - \partial_{xx})^{-\alpha}\theta \quad (6.1)$$

Since (6.1) satisfies the maximum principle, we have

$$\|\theta(t)\|_{L^\infty} \leq \|\theta_0\|_{L^\infty}.$$

We also obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2}^2 + \left\| \Lambda^{\frac{\gamma}{2}} \theta \right\|_{L^2}^2 &= - \int u \theta_x \theta dx \leq \|\theta\|_{L^\infty} \|u_x\|_{L^2} \|\theta\|_{L^2} \\ &\leq C (\|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^2) \|\theta\|_{L^2}^2 + \frac{1}{2} \left\| \Lambda^{\frac{\gamma}{2}} \theta \right\|_{L^2}^2 \end{aligned}$$

where we use the condition  $\alpha = \frac{1}{2} - \frac{\gamma}{4}$  to bound  $u_x$  as

$$\|u_x\|_{L^2} \leq C \left( \|\theta\|_{L^2} + \left\| \Lambda^{\frac{\gamma}{2}} \theta \right\|_{L^2} \right).$$

Therefore, we obtain that

$$\|\theta(t)\|_{L^2}^2 + \int_0^t \left\| \Lambda^{\frac{\gamma}{2}} \theta(s) \right\|_{L^2}^2 ds \leq \|\theta_0\|_{L^2}^2 e^{C(1 + \|\theta_0\|_{L^\infty}^2)t}.$$

**6.2. Approximation and passing to limit.** We consider the following equation with regularized initial data:

$$\theta_t^\epsilon + u^\epsilon \theta_x^\epsilon + \Lambda^\gamma \theta^\epsilon = 0, \quad \theta_0^\epsilon = \rho_\epsilon * \theta_0. \quad (6.2)$$

Then, there exists a global-in-time smooth solution  $\theta^\epsilon$ . Moreover,  $\theta^\epsilon$  satisfies that

$$\|\theta^\epsilon(t)\|_{L^\infty} + \|\theta^\epsilon(t)\|_{L^2}^2 + \int_0^t \left\| \Lambda^{\frac{\gamma}{2}} \theta^\epsilon(s) \right\|_{L^2}^2 ds + \epsilon \|\nabla \theta^\epsilon\|_{L^2}^2 \leq \|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^2}^2 e^{C(1+\|\theta_0\|_{L^\infty}^2)t}. \quad (6.3)$$

Therefore,  $(\theta_\epsilon)$  is bounded in  $\mathcal{C}_T$  uniformly in  $\epsilon > 0$ . This implies the uniform bounds

$$u^\epsilon \in L^4(0, T; L^4), \quad \theta^\epsilon \in L^4(0, T; L^4), \quad u_x^\epsilon \in L^2(0, T; L^2).$$

Moreover, these bounds with the equation

$$\theta_t^\epsilon = -u^\epsilon \theta_x^\epsilon - \Lambda^\gamma \theta^\epsilon + \epsilon \theta_{xx}^\epsilon = -(u^\epsilon \theta^\epsilon)_x + u_x^\epsilon \theta^\epsilon - \Lambda^\gamma \theta^\epsilon + \epsilon \theta_{xx}^\epsilon$$

we also have that

$$\theta_t^\epsilon \in L^{\frac{4}{3}}(0, T; H^{-2}).$$

We now extract a subsequence of  $(\theta^\epsilon)$ , using the same index  $\epsilon$  for simplicity, and a function  $\theta \in \mathcal{C}_T$  and  $u = (1 - \partial_{xx})^{-\alpha} \theta$  such that

$$\begin{aligned} \theta^\epsilon &\xrightarrow{*} \theta \quad \text{in } L^\infty(0, T; L^p) \quad \text{for all } p \in (1, \infty), \\ \theta^\epsilon &\rightharpoonup \theta \quad \text{in } L^2\left(0, T; H^{\frac{\gamma}{2}}\right), \\ \theta^\epsilon &\rightarrow \theta \quad \text{in } L^2(0, T; L_{\text{loc}}^p) \quad \text{for all } p \in (1, \infty), \\ u_x^\epsilon &\rightharpoonup u_x \quad \text{in } L^2(0, T; L^2), \end{aligned} \quad (6.4)$$

where we use Lemma 2.4 to obtain the strong convergence.

We now multiply (6.2) by a test function  $\psi \in \mathcal{C}_c^\infty([0, T] \times \mathbb{R})$  and integrate over  $\mathbb{R}$ . Then,

$$\int_0^T \int [\theta^\epsilon \psi_t + u^\epsilon \theta^\epsilon \psi_x + u_x^\epsilon \theta^\epsilon \psi - \theta^\epsilon \Lambda^\gamma \psi + \epsilon \theta^\epsilon \psi_{xx}] dx dt = \int \theta_0^\epsilon(x) \psi(0, x) dx.$$

By Lemma 2.4 with

$$X_0 = L^2\left(0, T; H^{\frac{\gamma}{2}}\right), \quad X = L^2(0, T; L_{\text{loc}}^2), \quad X_1 = L^2(0, T; H^{-2})$$

and using (6.4), we can pass to the limit to obtain that

$$\int_0^T \int [\theta \psi_t + u \theta \psi_x + u_x \theta \psi - \theta \Lambda^\gamma \psi] dx dt = \int \theta_0(x) \psi(0, x) dx \quad (6.5)$$

This completes the proof.

## 7. PROOF OF THEOREM 1.5

**7.1. A priori estimates.** We consider the equation

$$\theta_t + u \theta_x + \Lambda \theta = 0, \quad u = (1 - \partial_{xx})^{-\frac{1}{4}} \theta \quad (7.1)$$

We multiply (7.1) by  $\theta w$  and integrate in  $x$ . Then,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2(wdx)}^2 + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 &= - \int u \theta_x \theta w dx - \int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx \\ &= \frac{1}{2} \int u_x \theta^2 w dx + \frac{1}{2} \int u \theta^2 w_x dx - \int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx. \end{aligned}$$

As in the proof of Theorem 1.2, we bound the commutator term as

$$\int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx \leq \frac{1}{4} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 + C \|\theta\|_{L^2(wdx)}^2.$$



To estimate terms involving  $u$ , we use  $\theta = (1 - \partial_{xx})^{\frac{1}{4}}u$  and (2.8) to obtain that

$$\begin{aligned} \int u_x \theta^2 w dx + \int u \theta^2 w_x dx &\leq C \|\theta_0\|_{L^\infty} \left( \|u\|_{L^2(wdx)} + \|u_x\|_{L^2(wdx)} \right) \|\theta\|_{L^2(wdx)} \\ &\leq C \|\theta_0\|_{L^\infty} \left( \|\theta\|_{L^2(wdx)} + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)} \right) \|\theta\|_{L^2(wdx)} \\ &\leq C \left( \|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^2 \right) \|\theta\|_{L^2(wdx)}^2 + \frac{1}{4} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2. \end{aligned}$$

Collecting all terms together, we obtain that

$$\frac{d}{dt} \|\theta\|_{L^2(wdx)}^2 + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 \leq C \left( \|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^2 \right) \|\theta\|_{L^2(wdx)}^2$$

and hence that

$$\|\theta(t)\|_{L^2(wdx)}^2 + \int_0^t \left\| \Lambda^{\frac{1}{2}} \theta(s) \right\|_{L^2(wdx)}^2 \leq \|\theta_0\|_{L^2(wdx)}^2 \exp \left[ C \left( \|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^2 \right) t \right]. \quad (7.2)$$

We next multiply (7.1) by  $-(\theta_x w)_x$  and integrate in  $x$ . Then,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta_x\|_{L^2(wdx)}^2 + \left\| \Lambda^{\frac{3}{2}} \theta \right\|_{L^2(wdx)}^2 &= \int u \theta_x (\theta_x w)_x dx + \int \Lambda^{\frac{3}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \Lambda \theta dx \\ &= -\frac{1}{2} \int u_x (\theta_x)^2 w dx + \frac{1}{2} \int u (\theta_x)^2 w_x dx - \int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx + \int \Lambda \theta \theta_x w_x dx. \end{aligned}$$

Following the computation in [18],

$$\begin{aligned} \frac{d}{dt} \|\theta_x\|_{L^2(wdx)}^2 + \left\| \Lambda^{\frac{3}{2}} \theta \right\|_{L^2(wdx)}^2 &\leq C \|u\|_{H^1(wdx)} \|\theta_x\|_{L^4(wdx)}^2 + C \|\theta_x\|_{L^2(wdx)}^2 + \frac{1}{4} \left\| \Lambda^{\frac{3}{2}} \theta \right\|_{L^2(wdx)}^2 \\ &\leq C \|\theta\|_{H^{\frac{1}{2}}(wdx)} \|\theta_x\|_{L^4(wdx)}^2 + C \|\theta_x\|_{L^2(wdx)}^2 + \frac{1}{4} \left\| \Lambda^{\frac{3}{2}} \theta \right\|_{L^2(wdx)}^2, \end{aligned}$$

where we use the relation in (2.8) to bound  $u$  in terms of  $\theta$ . Since

$$\begin{aligned} \|\theta\|_{H^{\frac{1}{2}}(wdx)} \|\theta_x\|_{L^4(wdx)}^2 &\leq C \|\theta\|_{H^{\frac{1}{2}}(wdx)} \|\theta_x\|_{L^2(wdx)} \left\| \Lambda^{\frac{3}{2}} \theta \right\|_{L^2(wdx)} \\ &\leq C \|\theta\|_{H^{\frac{1}{2}}(wdx)}^2 \|\theta_x\|_{L^2(wdx)}^2 + \frac{1}{4} \left\| \Lambda^{\frac{3}{2}} \theta \right\|_{L^2(wdx)}^2 \end{aligned} \quad (7.3)$$

by (2.9), we obtain that

$$\frac{d}{dt} \|\theta_x\|_{L^2(wdx)}^2 + \left\| \Lambda^{\frac{3}{2}} \theta \right\|_{L^2(wdx)}^2 \leq C \left( 1 + \|\theta\|_{H^{\frac{1}{2}}(wdx)}^2 \right) \|\theta_x\|_{L^2(wdx)}^2. \quad (7.4)$$

Integrating in time (7.4) and using (7.2), we obtain that

$$\begin{aligned} \|\theta_x(t)\|_{L^2(wdx)}^2 + \int_0^t \left\| \Lambda^{\frac{3}{2}} \theta(s) \right\|_{L^2(wdx)}^2 &\leq \|\theta_{0x}\|_{L^2(wdx)}^2 \exp \left[ \int_0^t C \left( 1 + \|\theta(s)\|_{H^{\frac{1}{2}}(wdx)}^2 \right) ds \right] \\ &\leq \|\theta_{0x}\|_{L^2(wdx)}^2 \exp \left[ Ct + \|\theta_0\|_{L^2(wdx)}^2 \exp \left[ C \left( \|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^2 \right) t \right] \right]. \end{aligned} \quad (7.5)$$

By (7.2) and (7.5), we finally obtain that

$$\begin{aligned} \|\theta(t)\|_{H^1(wdx)}^2 + \int_0^t \left\| \Lambda^{\frac{1}{2}} \theta(s) \right\|_{H^1(wdx)}^2 &\leq C \|\theta_0\|_{H^1(wdx)}^2 \exp \left[ Ct + \|\theta_0\|_{L^2(wdx)}^2 \exp \left[ C \left( \|\theta_0\|_{L^\infty} + \|\theta_0\|_{L^\infty}^2 \right) t \right] \right]. \end{aligned} \quad (7.6)$$

**7.2. Approximation and passing to limit.** Since  $\theta$  is more regular than a solution in Theorem 1.2, we can follow the procedure in the proof of Theorem 1.2.

**7.3. Uniqueness.** To show the uniqueness of a weak solution, we consider the equation of  $\theta = \theta_1 - \theta_2$  given by

$$\theta_t + \Lambda\theta = -u_1\theta_x + u\theta_{2x}, \quad \theta(0, x) = 0. \quad (7.7)$$

We multiply  $w\theta$  to (7.7) and integrate over  $\mathbb{R}$ . Then,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2(wdx)}^2 + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 &= \int [-u_1\theta_x + u\theta_{2x}] \theta w dx - \int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx \\ &= \frac{1}{2} \int u_{1x} \theta^2 w dx + \frac{1}{2} \int u_1 \theta^2 w_x dx + \int u \theta_{2x} \theta w dx - \int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx. \end{aligned}$$

As before, the last term is bounded by

$$\int \Lambda^{\frac{1}{2}} \theta \left[ \Lambda^{\frac{1}{2}}, w \right] \theta dx \leq \frac{1}{2} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 + C \|\theta\|_{L^2(wdx)}^2.$$

The first three terms in the right-hand side are easily bounded by

$$C \left( \|\theta_{2x}\|_{L^2(wdx)} + \|\theta_1\|_{H^{\frac{1}{2}}(wdx)} \right) \left( \|\theta\|_{L^4(wdx)}^2 + \|\theta\|_{L^4(wdx)} \|u\|_{L^4(wdx)} \right).$$

By (2.9), we obtain that

$$\frac{d}{dt} \|\theta\|_{L^2(wdx)}^2 + \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2 \leq C \left( 1 + \|\theta_2\|_{H^1(wdx)}^2 + \|\theta_1\|_{H^{\frac{1}{2}}(wdx)}^2 \right) \|\theta\|_{L^2(wdx)}^2 + \frac{1}{2} \left\| \Lambda^{\frac{1}{2}} \theta \right\|_{L^2(wdx)}^2.$$

Since

$$\theta_2 \in L^2(0, T : H^1(wdx)), \quad \theta_1 \in L^2\left(0, T : H^{\frac{1}{2}}(wdx)\right),$$

we conclude that  $\theta = 0$  in  $L^2(wdx)$  and thus a weak solution is unique. This completes the proof of Theorem 1.5.

## 8. PROOF OF THEOREM 1.6

Taking the Fourier transform of (1.7), we have that

$$\partial_t |\hat{\theta}(\xi)| = -\operatorname{Re} \left[ \int_{\mathbb{R}} \frac{1}{(1+|\zeta|^2)^\alpha} \hat{\theta}(\zeta) i(\xi - \zeta) \hat{\theta}(\xi - \zeta) d\zeta \frac{\bar{\hat{\theta}}(\xi)}{|\hat{\theta}(\xi)|} \right] \frac{1}{\sqrt{2\pi}} - |\xi| |\hat{\theta}|.$$

Consequently, ignoring the factor  $\frac{1}{(1+|\zeta|^2)^\alpha}$ , we follow the proof of Theorem 1.3 with  $\delta = 0$  and the smallness condition (1.8) to obtain that

$$\|\theta(t)\|_{A^1} + \left( 1 - \frac{\sqrt{2}\|\theta_0\|_{A^0}}{\sqrt{\pi}} \right) \int_0^t \|\theta_x(s)\|_{A^1} ds \leq \|\theta_0\|_{A^1} \quad (8.1)$$

for all  $t \geq 0$ . We also follow the proof of Theorem 1.3 to obtain a unique weak solution via the approximation procedure. This completes the proof.

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